International Journal of Theoretical Physics, Vol. 23, No. 6, 1984

Gravitation as an Averaged Theory

M. N. Mahanta

Department of Mathematics, Salahaddin University, Erbil, Iraq

Received September 30, 1982

A result due to Eisenhart is used to justify an averaging process of a theory of gravity. More results of the theory are also deduced, similar to those derived by Einstein for the Nordström theory.

In a recent work (Mahanta, 1980), which will be referred to as I, an attempt was made to derive the Einstein field equations of general relativity by a kind of averaging procedure from the field equations of a model (Mahanta, 1979) proposed to represent hadronic interactions. This model uses a conformally flat space-time and has certain features common with the Nordström theory (Nordström, 1913) but is more general since in addition to the familiar scalar field another fourth-rank tensor P^{ijkl} seems to play an important part. However, the basic field equation in Nordström theory also appears in this model (Mahanta, 1979) as

$$R = \frac{12K}{a} T \tag{1}$$

or after writing $a = Kc^4/\pi G$ from I:

$$R = (12\pi G/c^4)T \tag{2}$$

The Einstein field equations are supposed to result from an averaging procedure (over space-times of atomic dimensions and periods) from the following equation:

$$G^j_k \equiv R^j_k - \tfrac{1}{2}\delta^i_k R = -(8\pi G/c^4)\left[T^j_k + \frac{1}{K}\left(-P^{ij\,l}_{k;\,il} + \Pi^j_k\right)\right] \tag{3}$$

As a result of this averaging we made the assumption that the tensor

$$T_k^j + \frac{1}{K}\left(-P_{k;il}^{ij;l} + \Pi_k^j\right)$$

manifests itself as the macroscopic energy-momentum tensor $T_{k(mac)}^j$, while the left-hand side

$$\bar{G}_k^j \equiv \bar{R}_k^j - \tfrac{1}{2}\delta_k^j\bar{R}$$

becomes the Einstein tensor in a V_4 whose fundamental tensor $g_{ij(mac)}$ is obtained by solving the Einstein field equations

$$\bar{R}_k^j - \tfrac{1}{2}\delta_k^j\bar{R} = -(8\pi G/c^4)T_{k(mac)}^j \tag{4}$$

In this paper we try to justify this assumption. We take the conformally flat metric of microscopic space-time as

$$ds^2 = H^2\left[(dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2\right] \tag{5}$$

so that

$$g_{ij} = H^2\eta_{ij} \qquad \text{and} \qquad g^{ij} = \frac{1}{H^2}\eta^{ij} \tag{6}$$

η_{ij} being the Minkowski tensor. The components of the Ricci tensor in covariant, contravariant, and mixed forms are easily calculated (Eisenhart, 1966) for the metric (5). In particular,

$$\begin{aligned} R \equiv g_{ij}R^{ij} &= \frac{6}{H^3}\left[\frac{\partial^2 H}{(\partial x^0)^2} - \frac{\partial^2 H}{(\partial x^1)^2} - \frac{\partial^2 H}{(\partial x^2)^2} - \frac{\partial^2 H}{(\partial x^3)^2}\right] \\ &\equiv \left(\frac{6}{H^3}\right)\Box H \end{aligned} \tag{7}$$

Thus equation (2) becomes the Nordström equation (Einstein, 1914)

$$H\Box H = (2\pi G/c^4)T(-g)^{1/2} \tag{8}$$

Our considerations will be based on (i) the following result due to Eisenhart (1966) and (ii) an approximation used by Einstein (1913) in his critique of the Nordström theory.

(i) If for a V_4, $/R_{ij} - \rho g_{ij}/ = 0$ admits of a simple root ρ_0 and a triple root ρ_1, the elementary divisors being simple, the principal directions corresponding to ρ_0 and ρ_1 satisfy the respective conditions,

$$g_{ij}\lambda^i_{0/}\lambda^j_{0/}=1, \qquad g_{ij}\lambda^i_{h/}\lambda^j_{h/}=-1 \qquad (h=1,2,3)$$

then

$$R_{ij} - \tfrac{1}{2}g_{ij}R = (\rho_0 - \rho_1)\lambda_{0/i}\lambda_{0/j} - \tfrac{1}{2}(\rho_0 + \rho_1)g_{ij} \tag{9}$$

representing a continuum of a perfect fluid, the congruence $\lambda^j_{0/}$ consisting of the lines of flow.

(ii) The space-time regions over which the averaging is performed are so small that the quantities $(\ln H)_{,i} \equiv (1/H)H_{,i}$ may be considered to be approximately constants over them (this assumption was made by Einstein to establish that in Nordström theory the gravitating mass of an isolated particle is determined by its inertial mass).

From considerations of symmetry and invariance of eigenvalues the assumption (i) is valid for the metric (5), ρ_0 belonging to a timelike congruence $\lambda_{0/i}$ and $\rho_1 = \rho_2 = \rho_3$ to three spacelike congruences, and it may be verified that under assumption (ii) the four-vector $H_{,i}$ is the eigenvector of the Ricci tensor R_{ij} with the nondegenerate eigenvalue

$$\rho_0 = (1/H^3)\Box H - (1/H^4)\eta^{ij}H_{,i}H_{,j} \tag{10}$$

Since $H_{,i}$ is timelike, we have

$$\lambda_{0/i} = (1/\Delta_1 H)^{1/2} H_{,i} \tag{11}$$

where

$$\Delta_1 H = g^{ij}H_{,i}H_{,j} = (1/H^2)\eta^{ij}H_{,i}H_{,j} > 0 \tag{12}$$

$\Delta_1 H$ is treated as a constant approximately over the space-times of atomic dimensions using assumption (ii) since it is formed from the $(1/H)H_{,i}$.

Writing equation (9) in the mixed form we get

$$G^j_k \equiv R^j_k - \tfrac{1}{2}\delta^j_k R = (\rho_0 - \rho_1)\lambda_{0/k}\lambda^j_{0/} - \tfrac{1}{2}(\rho_0 + \rho_1)\delta^j_k \tag{13}$$

$$= (\rho_0 - \rho_1)(1/H^2\Delta_1 H)\eta^{jl}H_{,k}H_{,l} - \tfrac{1}{2}(\rho_0 + \rho_1)\delta^j_k \tag{14}$$

using equation (11). The space-time averaging over a small region applied to equation (14) gives

$$\bar{G}^j_k \equiv \bar{R}^j_k - \tfrac{1}{2}\delta^j_k \bar{R} = (\bar{\rho}_0 - \bar{\rho}_1)(1/H^2\Delta_1 H)\eta^{jl}H_{,k}H_{,l} - \tfrac{1}{2}(\bar{\rho}_0 + \bar{\rho}_1)\delta^j_k \quad (15)$$

using assumption (ii), i.e.,

$$\bar{G}^j_k = (\bar{\rho}_0 - \bar{\rho}_1)\lambda_{0/k}\lambda^j_{0/} - \tfrac{1}{2}(\bar{\rho}_0 + \bar{\rho}_1)\delta^j_k \quad (16)$$

The raising of the suffix j in the first term on the right-hand side is by the tensor g^{jl} but we can show that $\lambda_{0/k}$ and $\lambda^j_{0/}$ can also be regarded as the covariant and contravariant forms of the principal congruence of the tensor $\bar{R}^j_k$ formed from the metric tensor $g_{ij(mac)}$ with the corresponding eigenvalue $\bar{\rho}_0$, the space-time average of ρ_0. To do so, consider the eigenvector equation of R^j_k for the eigenvalue ρ_0

$$(R^j_k - \delta^j_k \rho_0)\left[1/H(\Delta_1 H)^{1/2}\right]H_{,j} = 0 \quad (17)$$

Under assumption (ii) averaging of equation (17) over atomic space-times gives

$$(\bar{R}^j_k - \delta^j_k \bar{\rho}_0)\lambda_{0/j} = 0 \quad (18)$$

and

$$(\bar{R}^j_k - \delta^j_k \bar{\rho}_0)\lambda^k_{0/} = 0 \quad (19)$$

can be similarly obtained by averaging the equivalent form of the eigenvector equation

$$(R^j_k - \delta^j_k \rho_0)\lambda^k_{0/} = 0 \quad (20)$$

From equations (18) and (19) we see that $\lambda_{0/i}$, $\lambda^k_{0/}$ can also be regarded as the covariant and contravariant forms of the principal congruence of $\bar{R}^j_k$ for the eigenvalue $\bar{\rho}_0$ with $g_{ij(mac)}$ as the metric tensor used for raising and lowering of suffixes since equation (19) can also be obtained from equation (18) by using the tensor $g_{ij(mac)}$ on which $\bar{R}^j_k$ is based for raising and lowering of indices (for a nonrepeated eigenvalue like ρ_0 the eigenvector $\lambda^k_{0/}$ is uniquely determined).

Thus we see that the averaged Einstein tensor $\bar{G}^j_k$ is expressed by a formula (16) where the relevant quantities have meanings similar to those in (13) but with a changed metric, and we thus see that the tensor $\bar{G}^j_k$ retains both its tensor character and geometrical meaning after the averaging over atomic space-times as envisaged in I. This completes the first part of the work.

We add a few more results of our formalism.

We start with the equations

$$T^i_{k;j} = 0 \tag{21}$$

$$\text{i.e.,}\quad \left[T^j_k(-g)^{1/2}\right]_{,j} = \tfrac{1}{2}g_{jm,k}\left[T^{jm}(-g)^{1/2}\right] = (1/H)H_{,k}T(-g)^{1/2} \tag{22}$$

Using equation (8) we get

$$\left[T^j_k(-g)\right]_{,j} = (c^4/2\pi G)H_{,k}\Box H \tag{23}$$

If we put

$$t_{ij} = (c^4/2\pi G)\left[-H_{,i}H_{,j} + \tfrac{1}{2}g_{ij}(g^{mn}H_{,m}H_{,n})\right]$$

$$\text{i.e.,}\quad t^j_k = (c^4/2\pi G)(1/H^2)\left[-\eta^{jl}H_{,k}H_{,l} + \tfrac{1}{2}\delta^j_k(\eta^{mn}H_{,m}H_{,n})\right] \tag{24}$$

Then

$$(H^2 t^j_k)_{,j} = -(2\pi G/c^4)^{-1}H_{,k}\Box H \tag{25}$$

so that we get from equation (23)

$$\left[\left(T^j_k + \frac{1}{H^2}t^j_k\right)(-g)^{1/2}\right]_{,j} = 0 \tag{26}$$

giving an exact conservation equation (Einstein, 1913). The eigenvalue ρ_0 of the principal congruence $\lambda_{0/i} = (1/\Delta_1 H)^{1/2}H_{,i}$ (an approximate result) given by equation (10) can now be written as

$$\rho_0 = (2\pi G/c^4)(T - t/H^2) \tag{27}$$

Also from the relation (Eisenhart, 1966)

$$\rho_0 + \rho_1 + \rho_2 + \rho_3 = R \tag{28}$$

we have

$$\rho_1 = \tfrac{1}{3}(R - \rho_0) = (2\pi G/3c^4)(5T + t/H^2) \tag{29}$$

using relations (2) and (27).

ACKNOWLEDGMENT

The author would like to thank the authorities of the Salahaddin University for the hospitality and facilities for the work.

REFERENCES

Einstein, A. (1913). *Physikalische Zeitschrift*, **14**, 1249.
Einstein, A. (with A. D. Fokker) (1914). *Annalen der Physik (Leipzig)*, **44**, 321.
Eisenhart, L. P. (1966). *Riemannian Geometry*, pp. 119, 140, 114. Princeton University Press, Princeton, New Jersey.
Mahanta, M. N. (1979). *Lettere al Nuovo Cimento*, **26**, 317.
Mahanta, M. N. (1980). *Lettere al Nuovo Cimento*, **29**, 203.
Nordström, G. (1913). *Annalen der Physik (Leipzig)*, **42**, 533.